APPENDIX

Brief Review of Vector Algebra

A.0 Introduction

Vector algebra is used extensively in computational mechanics. The student must thus understand the concepts associated with this subject. The current review of vector algebra is not intended to be exhaustive, but only to present some concepts applicable to computational mechanics.

• **Definition** : scalar quantity

A scalar quantity is one possessing only magnitude; scalars add algebraically.

• **Definition** : vector quantity

A vector quantity is one possessing both magnitude and direction; vectors add according to the Parallelogram Law (or to the Triangle Rule). Geometrically a vector is represented by a directed line segment.

♣ **Notation :** To distinguish them from scalars, vectors will be denoted by lowercase symbols; i.e., the vector v is represented by either of the symbols { v } , **v** , or $\underset{\sim}{v}$.

A.1 Types of Vectors

• **Definition** : unit vector

Any vector whose magnitude is equal to one is called a unit vector.

• **Definition** : zero vector

Any vector whose magnitude is equal to zero is called a zero or null vector and is denoted by **0** or { 0 }.

• **Definition** : resultant

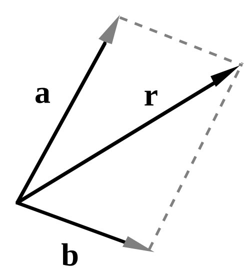
The resultant of a system of (two or more) vectors is the single vector which will replace the system and have the same effect on a body as the original system.

A.2 Vector Addition

For simplicity the subject of vector addition is presented in conjunction with vectors in the plane. The concepts, however, apply equally in three-dimensional space.

• Vector Addition Using the Parallelogram Law

The resultant **r** of two vectors **a** and **b** is the diagonal of the parallelogram for which **a** and **b** are adjacent sides. In symbols



$$\mathbf{r} = \mathbf{a} + \mathbf{b}$$

Figure A.1

• Vector Addition Using the Triangle Rule

To determine the resultant of vectors **a** and **b**, begin by placing the tail of one vector at the tip (arrow end) of the other. The resultant is then obtained by drawing a line from the tail end of the first vector to the tip of the second.

$$\mathbf{r} = \mathbf{a} + \mathbf{b}$$

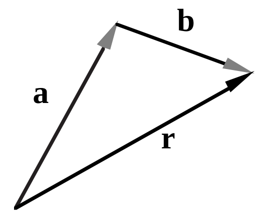


Figure A.2

- Comment : The resulting triangle comprises one half of the parallelogram shown above; from this we see the relation between the two approaches.

- Comment : Vector addition is *commutative*; i.e., $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

- Comment : Vector addition is *associative*; i.e., $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

A.3 Difference of Two Vectors

The difference **d** between the vectors **a** and **b** is obtained by adding the *negative* of **b** to **a**; i.e.,

$$\mathbf{a} + (-\mathbf{b}) = \mathbf{a} - \mathbf{b} \quad \text{(A.1)}$$

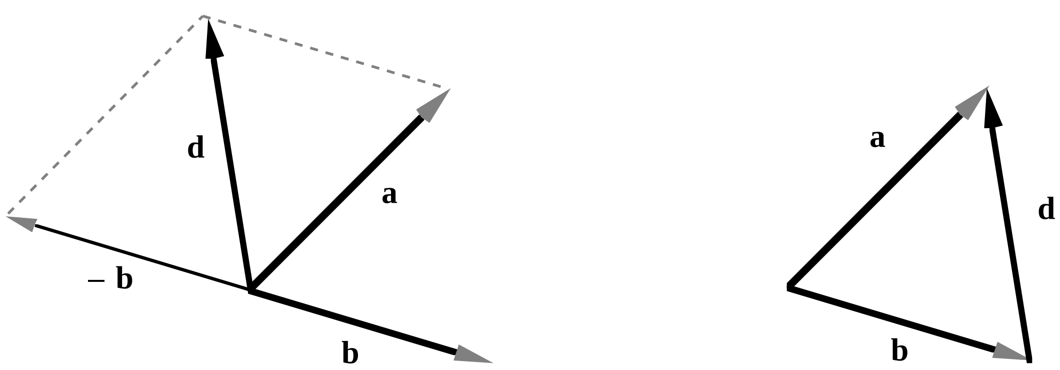


Figure A.3

A.4 Product of a Scalar and a Vector

The product of a scalar k and a vector $\mathbf{v}$ is the vector $k\mathbf{v}$ whose magnitude and direction are determined by the magnitude and sign of k. For example:

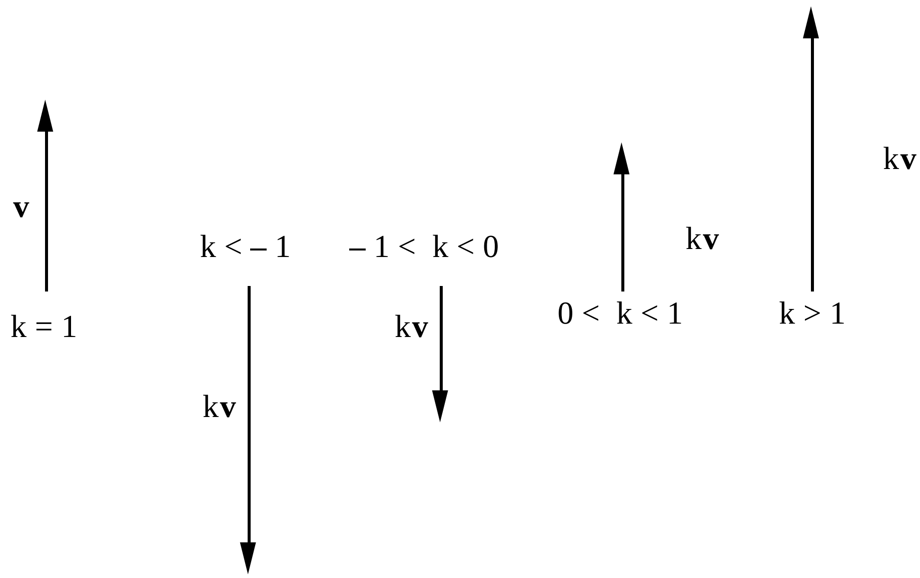


Figure A.4

If m and n represent scalars and $\mathbf{v}$ and $\mathbf{w}$ represent vectors, it follows that

$$(m + n)\,\mathbf{v} = m\,\mathbf{v} + n\,\mathbf{v} \tag{A.2}$$

$$m\,(\mathbf{v} + \mathbf{w}) = m\mathbf{v} + m\mathbf{w} \tag{A.3}$$

$$m\,(n\mathbf{v}) = n\,(m\mathbf{v}) = mn\mathbf{v} \tag{A.4}$$

A.5 Rectangular Components of a Vector

For the right-handed rectangular Cartesian coordinate system shown, denote the angles which a vector makes with the positive x-, y-, and z-axes by θ_x, θ_y, and θ_z, respectively. Since these angles are always measured from the positive side of the axes, they will always lie in the range 0 θ 180 degrees.

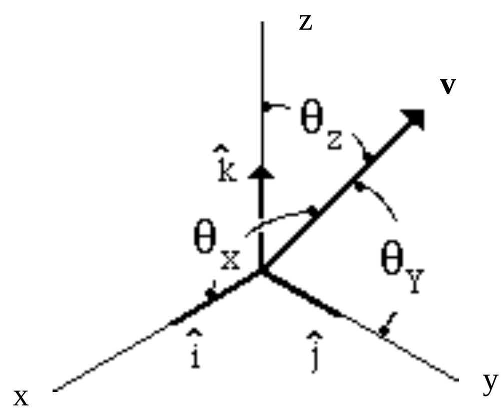


Figure A.5

The magnitude of a vector in *three* dimensions is given by

$$|\mathbf{v}| = v = \sqrt{(v_x)^2 + (v_y)^2 + (v_z)^2} \qquad \text{(A.5)}$$

where the *scalar components* or *direction numbers* of **v** are

$$v_x = v \cos\theta_x \;, \quad v_y = v \cos\theta_y \;, \quad v_z = v \cos\theta_z \qquad \text{(A.6)}$$

The angles θ_x, θ_y, and θ_z are called the *direction angles* of **v**; the quantities $\cos\theta_x$, $\cos\theta_y$, and $\cos\theta_z$ are called the *direction cosines* of **v** . Substituting the above equations into the expression for the magnitude of **v** gives

$$(\cos\theta_x)^2 + (\cos\theta_y)^2 + (\cos\theta_z)^2 = 1 \qquad \text{(A.7)}$$

which shows that *only two of the direction cosines are independent*.

The vector **v** may be written in terms of its scalar components in the following manner:

$$\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}} \qquad \text{(A.8a)}$$

$$= v\left(\cos\theta_x \hat{\mathbf{i}} + \cos\theta_y \hat{\mathbf{j}} + \cos\theta_z \hat{\mathbf{k}} \right) \qquad \text{(A.8b)}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ represent unit vectors parallel to the x, y and z-coordinate axes, respectively. The vector **v** is thus expressed as a product of a scalar (its magnitude v) and a unit vector directed along its line of action.

A.6 Addition of Vectors by Summing Rectangular Components

The resultant **r** of several vectors may be obtained by resolving the vectors into scalar components and then summing the respective components. This approach to vector addition is carried out in the following manner: For simplicity, consider the addition of only two vectors **a** and **b**. The approach used is, however, completely general and is easily extended to the case of more than two vectors. For the following vector sum

$$\mathbf{r} = \mathbf{a} + \mathbf{b} \tag{A.9}$$

express each vector in terms of scalar components multiplying unit vectors; i.e.,

$$r_x \hat{\mathbf{i}} + r_y \hat{\mathbf{j}} + r_z \hat{\mathbf{k}} = (a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}} + a_z \hat{\mathbf{k}}) + (b_x \hat{\mathbf{i}} + b_y \hat{\mathbf{j}} + b_z \hat{\mathbf{k}}) \tag{A.10a}$$

$$= (a_x + b_x) \hat{\mathbf{i}} + (a_y + b_y) \hat{\mathbf{j}} + (a_z + b_z) \hat{\mathbf{k}} \tag{A.10b}$$

From the above expressions it follows that

$$r_x = (a_x + b_x) \ , \ r_y = (a_y + b_y) \ , \ r_z = (a_z + b_z) \tag{A.11}$$

Next consider the exercise of summing n vectors. A typical vector is written in the form:

$$\mathbf{v}_m = v_{x_m} \hat{\mathbf{i}} + v_{y_m} \hat{\mathbf{j}} + v_{z_m} \hat{\mathbf{k}} \tag{A.12}$$

where $m = 1, 2, ..., n$. Extending the previous findings leads to the following relations:

$$r_x = \sum_{m=1}^{n} v_{x_m} \ , \ r_y = \sum_{m=1}^{n} v_{y_m} \ , \ r_z = \sum_{m=1}^{n} v_{z_m} \tag{A.13}$$

and

$$| \mathbf{r} | = r = \sqrt{(r_x)^2 + (r_y)^2 + (r_z)^2} \tag{A.14}$$

The directions cosines of **r** are given by:

$$\cos \theta_x = \frac{r_x}{r} \ , \ \cos \theta_y = \frac{r_y}{r} \ , \ \cos \theta_z = \frac{r_z}{r} \tag{A.15}$$

A.7 Scalar (Dot) Product

• **Definition** : scalar product

The scalar or dot product of two nonzero vectors **a** and **b** is a number given by

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\alpha = ab\cos\alpha \qquad \text{(A.16)}$$

where α represents the included angle between the two vectors **a** and **b** .

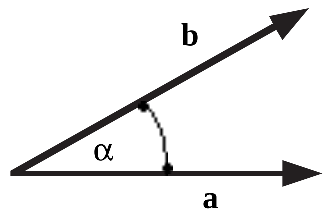


Figure A.6

The scalar product may be viewed as either:

i) the projection (a cos α) of **a** in the direction of **b** multiplied by the magnitude of **b** ; or
ii) the projection (b cos α) of **b** in the direction of **a** multiplied by the magnitude of **a**.

The following relations hold for scalar products:

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a} \qquad \text{(commutative law)}$$

$$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} \qquad \text{(distributive law)}$$

$$m(\mathbf{a} \bullet \mathbf{b}) = (m\mathbf{a}) \bullet \mathbf{b} = \mathbf{a} + (m\mathbf{b}) \qquad \text{(where "m" is a scalar)}$$

Since the unit vectors are *orthogonal* (i.e., oriented at right angles to each other), it follows that

$$\hat{\mathbf{i}} \bullet \hat{\mathbf{i}} = \hat{\mathbf{j}} \bullet \hat{\mathbf{j}} = \hat{\mathbf{k}} \bullet \hat{\mathbf{k}} = 1 \qquad \text{(A.17a)}$$

and that

$$\hat{\mathbf{i}} \bullet \hat{\mathbf{j}} = \hat{\mathbf{j}} \bullet \hat{\mathbf{k}} = \hat{\mathbf{k}} \bullet \hat{\mathbf{i}} = 0 \qquad \text{(A.17b)}$$

Consider the vectors

$$\mathbf{a} = (a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}}) \quad \text{and} \quad \mathbf{b} = (b_x\,\hat{\mathbf{i}} + b_y\,\hat{\mathbf{j}} + b_z\,\hat{\mathbf{k}}) \qquad \text{(A.18)}$$

Using the distributive law of scalar products in conjunction with the above relations between unit vectors, it follows that

$$\mathbf{a} \bullet \mathbf{b} = a_x b_x + a_y b_y + a_z b_z \qquad \text{(A.19)}$$

A.8 Vector (Cross) Product

The vector or cross product of two non-parallel vectors **a** and **b** is a vector **p** ; i.e.,

$$\mathbf{p} = \mathbf{a} \text{ x } \mathbf{b} \qquad \text{(A.20)}$$

The magnitude of **p** is equal to the product of the magnitudes of **a** and **b** and the sine of their included angle α ; i.e.,

$$|\mathbf{p}| = p = |\mathbf{a}||\mathbf{b}| \sin\alpha = ab \sin\alpha \qquad \text{(A.21)}$$

The line of action of **p** is perpendicular to the plane containing **a** and **b** (see Figure A.7).

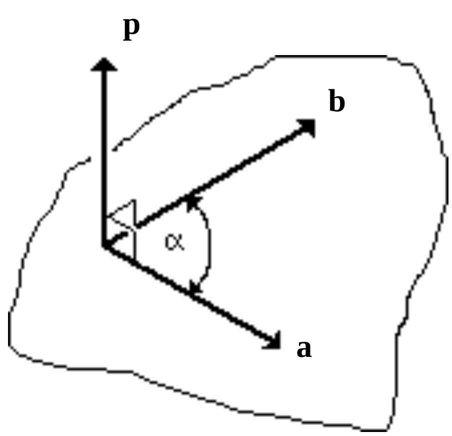


Figure A.7

The sense of **p** is given by the right hand rule. The geometric interpretation of the vector product is : the magnitude of the vector product **p** is equal to the area of the parallelogram which has **a** and **b** for sides.

The following relations hold for vector products:

$$\mathbf{a} \text{ x } (\mathbf{b} + \mathbf{c}) = \mathbf{a} \text{ x } \mathbf{b} + \mathbf{a} \text{ x } \mathbf{c} \qquad \text{(distributive law)}$$

$$m(\mathbf{a} \text{ x } \mathbf{b}) = (m\,\mathbf{a}) \text{ x } \mathbf{b} = \mathbf{a} \text{ x } (m\,\mathbf{b}) \qquad \text{(where "m" is a scalar)}$$

NOTE: vector products are *not* commutative; i.e., $\mathbf{a} \text{ x } \mathbf{b} \quad \mathbf{b} \text{ x } \mathbf{a}$

NOTE: vector products are *not* associative; i.e., $\mathbf{a} \text{ x } (\mathbf{b} \text{ x } \mathbf{c}) \quad (\mathbf{a} \text{ x } \mathbf{b}) \text{ x } \mathbf{c}$

Since the unit vectors are orthogonal (i.e., oriented at right angles to each other), it follows that

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0 \qquad \text{(A.22a)}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \qquad \text{(A.22b)}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \quad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}} \qquad \text{(A.22c)}$$

Consider the vectors

$$\mathbf{a} = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}} + a_z \hat{\mathbf{k}} \quad \text{and} \quad \mathbf{b} = b_x \hat{\mathbf{i}} + b_y \hat{\mathbf{j}} + b_z \hat{\mathbf{k}} \qquad \text{(A.23)}$$

Using the distributive law of vector products in conjunction with the above relations between unit vectors, it follows that

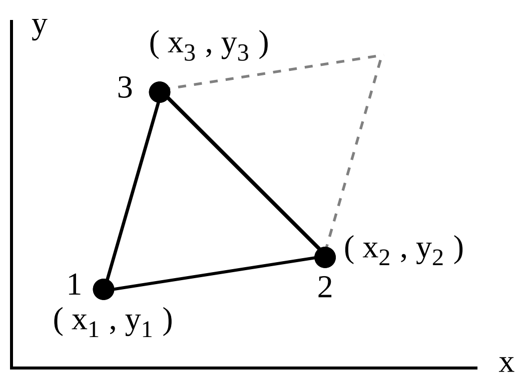
$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y) \hat{\mathbf{i}} + (a_z b_x - a_x b_z) \hat{\mathbf{j}} + (a_x b_y - a_y b_x) \hat{\mathbf{k}} \qquad \text{(A.24a)}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad \text{(A.24b)}$$

Thus, the value of the cross product of two vectors can be obtained by evaluating the above third-order determinant.

EXAMPLE A.1 : Application of the vector product to a triangular area

Consider the parallelogram associated with three co-planer points.



Let

$$\mathbf{v}_1 = (x_2 - x_1)\,\hat{\mathbf{i}} + (y_2 - y_1)\,\hat{\mathbf{j}} \qquad (1)$$

and

$$\mathbf{v}_2 = (x_3 - x_1)\,\hat{\mathbf{i}} + (y_3 - y_1)\,\hat{\mathbf{j}} \qquad (2)$$

The area 2A of the above parallelogram is equal to the cross product between $\mathbf{v}_1$ and $\mathbf{v}_2$; i.e.,

$$2A = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ (x_2 - x_1) & (y_2 - y_1) & 0 \\ (x_3 - x_1) & (y_3 - y_1) & 0 \end{vmatrix} \qquad (3a)$$

$$= \left[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) \right] \hat{\mathbf{k}} \qquad (3b)$$

$$= \left[x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) \right] \hat{\mathbf{k}} \qquad (3c)$$

Thus, the area of the triangular region is one-half of the above quantity. This result will be used in Chapter 9 in the discussion of interpolation functions associated with triangular elements.

♦

A.9 Scalar and Vector Fields

• **Definition** : gradient

The gradient of the function f = f (x, y, z) at the point (a, b, c) is defined to be

$$\nabla f = \frac{f}{x}\,\hat{\mathbf{i}} + \frac{f}{y}\,\hat{\mathbf{j}} + \frac{f}{z}\,\hat{\mathbf{k}} \qquad (A.25)$$

where each of the partial derivatives is evaluated at the point (a, b, c).

• **Definition** : vector field

A function which assigns a vector to each point in some region in the plane (or in space) is called a vector field.

• **Definition** : scalar field

A function which assigns a number to each point in some region in the plane (or in space) is called a scalar field.

• **Definition** : divergence

Let $\mathbf{f} = p\,\hat{\mathbf{i}} + q\,\hat{\mathbf{j}} + r\,\hat{\mathbf{k}}$ be a vector field in space. The scalar field

$$\frac{p}{x} + \frac{q}{y} + \frac{r}{z}$$

is called the divergence of **f** (also written as div **f**).

♣ **Notation** :

Let $\nabla \equiv \frac{}{x}\hat{\mathbf{i}} + \frac{}{y}\hat{\mathbf{j}} + \frac{}{z}\hat{\mathbf{k}}$. The divergence of **f** is thus written as

$$\nabla \bullet \mathbf{f} \equiv \frac{p}{x} + \frac{q}{y} + \frac{r}{z} \qquad \text{(A.26)}$$

• **Definition** : curl

Let $\mathbf{f} = p\,\hat{\mathbf{i}} + q\,\hat{\mathbf{j}} + r\,\hat{\mathbf{k}}$ be a vector field in space. The function which assigns to each point the vector

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{}{x} & \frac{}{y} & \frac{}{z} \\ p & q & r \end{vmatrix}$$

is called the curl of **f**. It is denoted by **curl f** or by ∇ x **f**.

♠ **Theorem** : Green's theorem

Let Ω be a convex region in the plane and let Γ be its boundary (swept out counter clockwise). Assume that the functions P and Q have continuous partial derivatives throughout Γ. Then

$$\int_{\Omega} \left(\frac{P}{x} + \frac{Q}{y} \right) dA = \oint_{\Gamma} P\,dy - Q\,dx \qquad \text{(A.27)}$$

♠ **Theorem** : Divergence theorem

Let Ω be a convex region in space and let Γ be its surface. Let **f** be a vector field in space. Then

$$\int_{\Omega} \left(\nabla \bullet \mathbf{f} \right) dV = \int_{\Gamma} \mathbf{f} \bullet \mathbf{n} \; ds \qquad \text{(A.28)}$$

where **n** represents the unit outward normal.

♠ **Theorem** : Stokes' theorem

Let S be part of the surface of a convex region in space and let C be its boundary curve. At each point of S let **n** be the unit outward normal to S. Let C be oriented by the right-hand rule. Let **f** be a vector field in space. Then

$$\int_{S} \left(\nabla \times \mathbf{f} \right) \bullet \mathbf{n} \, dA = \oint_{C} \mathbf{f} \bullet \mathbf{T} \; ds \qquad \text{(A.29)}$$

Stokes' theorem relates the tangential component **T** of a vector field along a closed curve to the *normal component* of the curl over a surface.

A.10 Mapping

• **Definition** : mapping

A mapping is defined as a function which assigns to a point in the plane a point in the plane; i.e.,

$$F(u, v) = (x, y) \qquad \text{(A.30)}$$

In Eq. (A.30) F represents the mapping function; (u, v) denotes the "input point"; and, (x, y) denotes the "output point".

• **Definition** : Jacobian

Let x = f (u, v) and y = g(u, v) describe a mapping from the u – v plane to the x – y plane. The function

$$\frac{f}{u}\frac{g}{v} - \frac{f}{v}\frac{g}{u} = \det\begin{bmatrix} \frac{f}{u} & \frac{f}{v} \\ \frac{g}{u} & \frac{g}{v} \end{bmatrix} \qquad \text{(A.31)}$$

is called the Jacobian of the mapping.

A.11 Exercises

Exercise A.1

Let **a** and **b** denote vectors and u and v denote scalars. Using indicial notation and tensor calculus, show that:

1) $\nabla (u + v) = \nabla u + \nabla v$ *or* grad (u + v) = grad u + grad v

2) $\nabla \cdot (\mathbf{a} + \mathbf{b}) = \nabla \cdot \mathbf{a} + \nabla \cdot \mathbf{b}$ *or* div (**a** + **b**) = div **a** + div **b**

3) $\nabla \times (\mathbf{a} + \mathbf{b}) = \nabla \times \mathbf{a} + \nabla \times \mathbf{b}$ *or* curl (**a** + **b**) = curl **a** + curl **b**

4) $\nabla \cdot (u\mathbf{a}) = (\nabla u) \cdot \mathbf{a} + u (\nabla \cdot \mathbf{a})$

5) $\nabla \times (u\mathbf{a}) = (\nabla u) \times \mathbf{a} + u (\nabla \times \mathbf{a})$

6) $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$

7) $\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\, \mathbf{a} - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}\, (\nabla \cdot \mathbf{b})$

8) $\nabla (\mathbf{a} \cdot \mathbf{b}) = (\mathbf{b} \cdot \nabla)\, \mathbf{a} + (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{b} \times (\nabla \times \mathbf{a}) + \mathbf{a} \times (\nabla \times \mathbf{b})$

9) $\nabla \cdot (\nabla u) = \nabla^2 u$

10) $\nabla \times (\nabla u) = 0$

11) $\nabla \cdot (\nabla \times \mathbf{a}) = 0$

12) $\nabla \times (\nabla \times \mathbf{a}) = \nabla (\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$

Exercise A.2

Given the vectors $\mathbf{a} = < 1, 3 >$ and $\mathbf{b} = < 3, 4 >$, calculate

a) $\mathbf{a} \cdot \mathbf{b}$ and $\cos(\mathbf{a}, \mathbf{b})$.

b) $\mathbf{a} \times \mathbf{b}$

Exercise A.3

Given $f = (x_1)^2 + (x_2)^2$, calculate $\nabla \mathbf{f}$. What is the divergence of the vector $\nabla \mathbf{f}$?

A.12 Suggested Reading

1. S. K. Stein, *Calculus and Analytic Geometry*, McGraw-Hill, New York, NY, 1977.